UDC 539.3

VARIATIONAL-ASYMPTOTIC METHOD OF CONSTRUCTING A THEORY OF SHELLS

PMM Vol. 43, No. 4, 1979, pp. 664-687

V. L. BERDICHEVSKII
(Moscow)

(Received June 20, 1978)

A method of investigating functionals containing a small parameter is proposed. A first approximation functional for a physically and geometrically nonlinear theory of shells, and functionals of refined theories of shells are constructed by this method by means of a functional of three-dimensional elasticity theory.

1. Variational-asymptotic method. In a certain set M of elements u let there be defined a functional $I(u, h)$ dependent on the small parameter h, and let us seek stationary points $I(u, h)$ in M. Simplification of this problem can be attempted as follows.

We construct a functional I_0 be discarding all small terms, in the asymptotic sense, in $I(u, h)$, and we let M_0 be the set of its stationary points. If the set M_0 consists of one element, then it is natural to expect that this element is a first approximation to one of the stationary points. However, the set M_0 contains many elements in the problems to be considered later, even when the initial functional has a single stationary point in M. We take some element $u_0 \in M_0$ and seek u in the form $u = u_0 + u'$, where u' is less than u_0 in the asymptotic sense. Let us keep just the principal terms in u', in the asymptotic sense, and the principal cross terms between u_0 and u' in the functional $I(u_0 + u', h)$. We then obtain the functional $I'(u_0, u', h)$. If the stationary point of the functional I' is unique, then u' is defined in terms of u_0 and h. In principle, however, arbitrariness can occur in the finding of u': $u' = u'(u_0, u_1, h)$, where u_1 runs through a certain set M_1. We later seek u in the form $u = u_0 + u'(u_0, u_1, h) + u''$, where u'' is less than u' in the asymptotic sense, and we find u'' in terms of u_0 and u_1 analogously to the preceding. As a rule, the arbitrariness vanishes starting with a certain step, and the whole solution is determined by a certain set of elements u_0, $u_1, \ldots, u_s$.

Let v denote the part of the expansion $u = u_0 + u' + u'' + \ldots$ in which new arbitrarinesses appear, and let N be the subset of M through which the elements v run when u_0 runs through M_0, u_1 runs through $M_1, \ldots$, and v' is the first correction to v: $u = v + v'$ (v' does not contain new arbitrariness and is defined in terms of v: $v' = v'(v, h)$).

Let us consider the problem of seeking stationary points of the functional $I(u, h)$ in N. In general, this problem is simpler than the initial problem since the set N is narrower than M. If it is "reasonable" (for example, the stationary point is found uniquely) and $I(u + v'(u, h), h)$ differs slightly from $I(u, h)$ in N, then it can be expected that the stationary points $I(u, h)$ in N, are a first approximation to the stationary points $I(u, h)$. If $I(u + v'(u, h), h)$ differs by a finite quantity from $I(u, h)$ in N, and taking account of v'' yields

a small contribution, then the stationary points of the functional $I(u + v'(u, h), h)$ yield the first approximation. Otherwise, the next approximation must be examined.

The set N can be guessed in many problems. Then the set M is represented as the sum $M = N + N'$ $(u = v + v', v \in N, v' \in N')$, v is fixed, small terms in the functional $I(v + v', h)$ are discarded, and the stationary point in v' is sought. Corrections to $v'(v, h)$ and functionals of the first and subsequent approximations are later constructed by the general scheme.

Sometimes a difficulty is met in the estimates, which is associated with the fact that the order of the desired functions is not known in advance, and it is not clear with respect to the separate components as to whether they must be retained in the approximation under consideration. In these cases, first similar terms should be discarded, the orders of the desired functions should be found, and then the orders of the discarded terms should be determined. If the orders of the discarded terms turn out to be less than those taken into account, then it can be considered that the approximation has been constructed correctly. Otherwise, they must be retained.

All terms of order $\varepsilon(h)$ as compared to one must be kept in the functional in order to construct a refined theory taking account of corrections of order $\varepsilon(h)$.

Let us clarify this assertion by the example of the problem of the minimum of a quadratic functional $I(u, h)$ of the form

$$I(u, h) = E(u, u, h) - L(u, h)$$
$$E(u, v, h) = E_0(u, v) + \varepsilon_1(h) E_1(u, v) + \varepsilon_2(h) E_2(u, v) + \ldots$$
$$L(u, h) = L_0(u) + \varepsilon_1(h) L_1(u) + \varepsilon_2(h) L_2(u) + \ldots, \quad \varepsilon_2 = o(\varepsilon_1)$$

in a linear space M. Here $E(u, v, h)$ is a symmetric functional, bilinear in u, v $L(u, h)$ is a linear functional, and $E_0(u, u)$ is a positive-definite functional.

The first approximation u_0 is a minimum point of the functional $I_0 = E_0(u, u) - L_0(u)$. The element u_0 satisfies the Euler equations: for any element w from M

$$2E_0(u_0, w) - L(w) = 0 \qquad (1.1)$$

Let us represent u in the form $u = u_0 + u'$. Keeping the principal term containing u' and the principal cross term between u_0 and u' in $I(u, h)$, and using the equality $2E_0(u_0, u') - L_0(u') = 0$, resulting from (1.1), we obtain the problem of the minimum of the functional

$$I' = E_0(u', u') + 2\varepsilon_1(h) E_1(u_0, u') - \varepsilon_1(h) L_1(u')$$

to determine u'. After the substitution $u' = \varepsilon_1(h) z$ it is reduced to the problem of the minimum of the functional $\varepsilon_1^{-2}(h) I' = E_0(z, z) + 2E_1(u_0, z) - L_1(z)$, which is independent of the small parameter. Therefore, $u' \sim \varepsilon_1(h)$ and terms of the order of $\varepsilon_1(h)$ must be retained in the functional to construct the solution to $\varepsilon_1(h)$ accuracy.

The variational-asymptotic method is used below on the problem of deriving a two-dimensional theory of shells from three-dimensional elasticity theory (of the papers devoted to this topic, we note [1–20]). The solution of the three-dimensional problem depends on a small parameter in an essentially different manner at the edge and in the interior of the shell. Hence, the energy functional is separated into the sum of two functionals, the edge and the interior functionals and its own iteration process is created for each. Consequently, the shell energy is comprised of energy distributed along the middle surface, and energy concentrated at the edge.

Four approximations are constructed: the classical theory having the error $\varepsilon + h_* + h_{**}$ (ε is the deformation scale, $h_* = h/R_0$, R_0 is the characteristic radius of curvature, $h_{**} = h/l$, l is the characteristic scale of the change in the state of stress), the refined theory in which terms of order h_{**} are conserved, the refined theory in which terms of order h_* and h_{**} are conserved, and the approximation in which corrections on the order of h_*, h_{**}, and h_{**}^2 are kept.

The classical theory is examined in the case of inhomogeneity, anisotropy, and physical nonlinearity, and the next approximations for homogeneous, isotropic, and physically linear shells.

The energy distributed over the middle surface differs in the refined theory from the classical expression by the inclusion of cross terms between the tension and bending tensors. In the approximation taking account of corrections on the order of h_{**}^2, the transverse shear energy and two additional desired functions describing the shear are added. In this respect it recalls the Reissner—Naghdi theory.

The edge energies can be neglected in the classical theory. Taking the edge energy into account is essential in the next approximations. The edge energy is calculated here in an approximation permitting corrections of order h_* and h_{**} to be taken into account. It is shown that the edge energy of a loaded edge depends on the self-equilibrated part of the load at higher approximations. As a rule, the self-equilibrated part of the load is unknown in applied problems, hence the edge energy is not determined and application of theories describing corrections on the order of h_{**}^2 is meaningless (theories taking account of transverse shear are referred to here). Closed shells, as well shells with free and rigidly fixed edges for which the edge energy is found uniquely are the exception.

The edge energy describes the influence of the boundary layer on the inner state of stress. It was studied first by Gol'denveizer [3]. The idea of an edge energy was expressed by Koiter and Heijden [16, 18]. From physical considerations they indicated the energy of the free edge of an isotropic shell. The true energy of the edge domain is less for a free edge than is the energy calculated by the two-dimensional theory of shells. Hence, the shell energy in [16, 18] was represented as a difference between the energies distributed over the whole middle surface (including the edge domain) and some positive integral over the middle surface boundary. The functional constructed in this manner turns out to be non-convex and without a lower bound. In this connection, it is possible to mention the problems which either have no solution or have an ambiguous solution (see Sect. 8).

In the theory considered below, the energy is composed of the sum of the energies from the inner parts of the shell and the energy of an edge rod. The appropriate problem is substantially the problem of connecting the inner part of the shell and an edge rod with identical elastic characteristics. This will permit elimination of the defect noted.

The connection problem results in edge conditions containing derivatives of the same order as the differential equations. Such boundary value problems occurred even under other approaches to the construction of refined shell theories [5].

Known at the present time, as other applications of the variational-asymptotic method, are the derivation of a geometrically nonlinear theory of anisotropic inhomogeneous rods [21], a refined linear theory of rods [22], a refined theory of low-

frequency shell vibrations, [23], a theory of high-frequency plate vibrations [24], and straight rods [25], the derivation of averaged equations describing the continuum with a periodic microstructure [26].

Different publications of the method are given in [21, 23] (*). The foundation of the method is not considered, however, for strictly convex functionals it is represented completely visibly.

2. Formulation of the problem. Let us consider a smooth surface Ω_0 in a three-dimensional space R (smoothness is understood to be the existence of derivatives of the order which will be encountered later). Let V_0 be the three-dimensional domain marked out by the vectors $^1/_2h\mathbf{n}_0$, $-^1/_2h\mathbf{n}_0$, where $\mathbf{n}_0$ is a vector normal to Ω_0. The elastic body (shell) occupies the domain V_0 in the undeformed state.

We introduce a curvilinear coordinate system ξ^α, ξ in V_0 by means of the formulas

$$x^i = r_0{}^i(\xi^\alpha) + \xi n_0{}^i(\xi^\alpha) \tag{2.1}$$

Here x^i are Cartesian coordinates in R, $x^i = r_0{}^i(\xi^\alpha)$ is the equation of the surface Ω_0, $n_0{}^i$ are components of the vector $\mathbf{n}_0$, the coordinates ξ^α, ξ vary in a cylinder of height h: $\xi^\alpha \in \Omega_0$, $|\xi| \leqslant h/2$, the domain of variation of the ξ^α, exactly as of the middle surface, is denoted by Ω_0. The coordinates ξ^α, ξ are also denoted by ξ^α, the small Latin superscripts $a, b, c, \ldots$ and $i, j, k, \ldots$ are ascribed to projections on the associated axis ξ^a and the observer axis x^i and run through the values $1, 2, 3$, the associated superscript 3 is sometimes omitted (for instance $\xi = \xi^3$), the small Greek superscripts run through the values $1, 2$ and correspond to projections on the axis ξ^α. The zero subscript denotes quantities in the undeformed state.

The metric tensor components in the coordinate system (2.1) are given by the formulas

$$\begin{aligned}
&g_{0\alpha\beta} = a_{0\alpha\beta} - 2\xi b_{0\alpha\beta} + \xi^2 c_{0\alpha\beta}, \quad g_{0\alpha 3} = 0, \quad g_{033} = 1 \\
&g_0^{\alpha\beta} = \varkappa^{-2}[(1-2H_0\xi)^2 a_0^{\alpha\beta} + 2\xi(1-2H_0\xi) b_0^{\alpha\beta} + \xi^2 c_0^{\alpha\beta}] \\
&g_0^{\alpha 3} = 0, \quad g_0^{33} = 1 \\
&\varkappa = 1 - 2H_0\xi + K_0\xi^2 = \sqrt{\det \| g_{\alpha\beta} \|}\,/\,a_0 \\
&(a_{0\alpha\beta} = r^i_{0\alpha} r_{0i\beta}, \quad a_0 = \det \| a_{0\alpha\beta} \|, \quad r^i_{0\alpha} \equiv r^i_{0,\alpha} \\
&b_{0\alpha\beta} = n_{0i} r^i_{0,\alpha\beta}, \quad c_{0\alpha\beta} = b^\lambda_{0\alpha} b_{0\lambda\beta} = -K_0 a_{0\alpha\beta} + 2H_0 b_{0\alpha\beta})
\end{aligned} \tag{2.2}$$

Here $a_{0\alpha\beta}$, $b_{0\alpha\beta}$, $c_{0\alpha\beta}$ are components of the first, second, and third quadratic forms of the surface, the comma in the subscripts denoted differentiation with respect to ξ^α, H_0 and K_0 are the mean and Gaussian curvatures of Ω_0. Unless specified

*) See also, Berdichevskii, V. L., Variational-asymptotic method of construction of a shell theory. Tezisy Dokl. XI All-Union Conf. on Plate and Shell Theory (Khar'kov, 1977). Moscow, 1977.

otherwise, juggling the surface subscripts is realized by using the metric $a_{0\alpha\beta}$. Summation is over repeated sub- and superscripts. Following [8], we shall understand the best constants in the inequalities

$$|b_{0\beta}^{\alpha}| \leqslant \frac{1}{R_0}, \quad |b_{0\beta,\gamma}^{\alpha}| \leqslant \frac{1}{R_0^2}$$

to be the characteristic radius of curvature R_0 of the surface Ω_0.

We select the coordinates ξ^1, ξ^2 in the neighborhood of the boundary Γ_0 of the surface Ω_0 so that the equation of Γ_0 would have the form $\xi^1 = c = \text{const}$ and ξ^1 would decrease with distance from Γ_0.

Let us consider that dead surface forces P_i are given on $S_0 = \Gamma_0 \times [-h/2, h/2]$ and $\Omega_{\pm}$ $(\xi = \pm h/2)$. Then the functions $x^i(\xi^\alpha, \zeta)$ governing the position of points of the body in the deformed state are stationary points of the functional [27]

$$I = \int_{V_0} U\varkappa \, d\omega \, d\xi - \int_{\Omega_0} \{P_i x^i(\xi^\alpha, \xi)\varkappa\} \, d\omega - \int_{S_0} P_i x^i(\xi^\alpha, \xi) \, d\sigma \qquad (2.3)$$

$$\{A\} \equiv A|_{\xi=h/2} + A|_{\xi=-h/2}, \quad d\omega = \sqrt{a_0}\, d\xi^1 d\xi^2$$

Here $d\omega$ is the area element on Ω_0, the elastic energy per unit volume U is a known function of the strain tensor components

$$2\varepsilon_{\alpha\beta} = x_{i,\alpha}x^i_{,\beta} - g_{0\alpha\beta}, \quad 2\varepsilon_{\alpha 3} = x_{i,\alpha}x^i_{\xi}, \quad 2\varepsilon_{33} = x_{i,\xi}x^i_{\xi} - 1 \qquad (2.4)$$

and the comma before the subscript ξ denotes differentiation with respect to ξ. The strain amplitude $\varepsilon = \max_{V_0}(\varepsilon_{ab}\varepsilon^{ab})^{1/2}$ is assumed small $(\varepsilon \ll 1)$.

It is required to replace the problem formulated by an approximate "two-dimensional" problem in which functions of just ξ^1, ξ^2 enter. The usual two-dimensional problem can be considered the result of passing to the limit as $h \to 0$ in which U and P_i depend on h in a known manner. The change of variable $\xi = h\zeta$, $|\zeta| \leqslant 1/2$ makes the domain of definition of the solution identical for different h. After the change in parameter, h will enter the functional explicitly in terms of $\varepsilon_{\alpha 3}$ and ε_{33}. Let us start the exposition from this viewpoint, however, we shall later turn to a more "realistic" viewpoint according to which h is fixed and the expansion is carried out over certain dimensionless small (but not infinitesimal) parameters.

The fact is that as $h \to 0$ the bending strains also tend to zero if the strained middle surface arrives at some limit position as $h \to 0$. Because of the physical nonlinearity, the material can behave differently for $\varepsilon \sim 10^{-2}$ or $\varepsilon \sim 10^{-3}$, and the formal passage to the limit $h \to 0$ can lead out of the physically interesting domain.

Rejection of the infinite smallness of the parameters generally requires refinement of the meaning of the symbols o, O, $\sim$, $\ll$. This question is not discussed later. Let us just note that the possibility of passing to the limit $h \to 0$ is restored in the physically linear case and all the concepts of asymptotic analysis acquire the usual meaning.

3. First approximation in the physical linear theory. Deformed middle surface. Let us introduce the function

$$r^i(\xi^\alpha) = \langle x^i(\xi^\alpha, \zeta)\rangle$$

where $\langle\cdot\rangle$ is the integral with respect to ζ in the limits $[-1/2, 1/2]$. The surface Ω given by the equations $x^i = r^i(\xi^\alpha)$ will be called the deformed middle surface. Deformation of the middle surface is described by the tensors $A_{\alpha\beta} = 1/2\,(a_{\alpha\beta} - a_{0\alpha\beta})$, $B_{\alpha\beta} = b_{\alpha\beta} - b_{0\alpha\beta}$. Let us consider the elongation amplitudie $\varepsilon_A = \max_{\Omega_0} (A_{\alpha\beta}A^{\alpha\beta})^{1/2}$ and the bending amplitude $\varepsilon_B = 1/2\,h \max_{\Omega_0} (B_{\alpha\beta}B^{\alpha\beta})^{1/2}$ to be small compared to unity.

Characteristic scales. We make a change in the desired functions

$$x^i(\xi^\alpha, \zeta) = r^i(\xi^\alpha) + hn^i(\xi^\alpha)\,\zeta + hy^i(\xi^\alpha, \zeta) \tag{3.1}$$

where n^i is the normal to Ω. According to the definition of r^i, the functions $y^i(\xi^\alpha, \zeta)$ satisfy the constraints

$$\langle y^i(\xi^\alpha, \zeta)\rangle = 0 \tag{3.2}$$

Formula (3.1) sets up a mutually one-to-one correspondence between all the functions $x^i(\xi^\alpha, \zeta)$ and the pairs $\{r^i(\xi^\alpha), y^i(\xi^\alpha, \zeta)\}$ in which the y^i are subject to the conditions (3.2).

Let

$$y_\alpha \equiv r_\alpha{}^i y_i \quad (r_\alpha{}^i \equiv r_{,\alpha}{}^i), \quad y \equiv n^i y_i$$

$$\Delta_\alpha \equiv \max_{V_0} |y_{\alpha,\zeta}|, \quad \Delta \equiv \max_{V_0} |y_{,\zeta}|$$

Because of the Werthinger inequality [28] $|y_\alpha| \leqslant \Delta_\alpha$, $|y| \leqslant \Delta$ and if $\Delta_\alpha = \Delta = 0$, then also $y_\alpha = y = 0$.

Let us examine a certain point on the surface Ω_0. For sufficiently small numbers l_1, l_2 at this point the following inequalities are valid

$$|A_{\alpha\beta,\gamma}| \leqslant \frac{\varepsilon_A}{l_\gamma}, \quad \frac{1}{2}h\,|B_{\alpha\beta,\gamma}| \leqslant \frac{\varepsilon_B}{l_\gamma} \tag{3.3}$$

$$\max_\zeta |y_{\alpha,\beta}| \leqslant \frac{\Delta_\alpha}{l_\beta}, \quad \max_\zeta |y_{,\alpha}| \leqslant \frac{\Delta}{l_\alpha}$$

We call the best constants l_1, l_2 in the inequalities (3.3) the characteristic scales of variation of the strain along the axes ξ^1, ξ^2. The scales l_1, l_2 depend on the points of the surface.

The nature of the state of stress is essentially related to the relative magnitude of h, l_1 and l_2. Let us divide the shell into three parts V_1, V_2 and V_3, where the ratio of h to one scale is much less than one in V_1, and is on the order of or greater than one to the other scale; $h/l_1 \ll 1$ $h/l_2 \ll 1$ in V_2, and $h/l_1 \gtrsim 1$, $h/l_2 \gtrsim 1$ in V_3. The domains V_1, V_2, and V_3 are cylinders in the variables ξ^α, ζ. Their intersection with Ω_0 is denoted by Ω_1, Ω_2, and Ω_3. The subscripts on domains V_1 and V_2 are selected in conformity with the fact that it is possible to pass from the three-dimensional problem to approximately the one-dimensional problem in V_1 (to a rod), and to the two-dimensional problem in V_2. The problem remains three-dimensional in V_3. The domains V_1, V_2 and V_3 depend on h.

Let us assume that the shell contains no domain V_3 but consists of the domains V_1 and V_2, where the domain V_1 abuts the edge of the shell and is given by the equation $c - b \leqslant \xi^1 \leqslant c$, $b = b(\xi^2, h)$, $b \to 0$ as $h \to 0$, while $h/l_1 \sim 1$

and $h / l_2 \ll 1$ in V_1.

The values of l_1 and l_2, and therefore, the relative arrangement of the domains V_1, V_2 and V_3 are determined by the external load, the kinematic boundary conditions, and the geometry of the surface Ω_0. The assumptions made implicitly introduce a number of constraints. For instance, the assumption of no domain V_3 eliminates external forces, whose characteristic scale of variation is on the order of the shell thickness, and also shells for which $R_0 \sim h$. The description of the constraints occurring are explicitly related closely to the construction of error estimates for the approximate theories and is a separate problem.

Let l denote a quantity equal to $\min\{l_1, l_2\}$ in Ω_2 and l_2 in Ω_1. Such a nature of the change in the external load can be selected in the passage to the limit $h \to 0$ that the parameter $h_{**} = h / l$ will be a function of the geometric parameter $h_* = h / R_0$. In a number of papers, the dependence $h_{**} = (h_*)^{1-\theta}$ is taken essentially, where $0 \leqslant \theta < 1$ is the index of variability. The postulation of such a dependence is not related to the crux of the matter: the passage from the three-dimensional to the two-dimensional theory is possible even for other dependences, for instance, $h_{**} = (h_*)^{1-\theta} \ln h_*$, and in those cases when the parameter h_{**} is not related to h_* at all, it is only important that $h_{**} \ll 1$. The parameters h_* and h_{**} are later considered independent. Conformity with the terminology ordinarily used can be achieved if the equality $h_{**} = (h_*)^{1-\theta}$, is considered the definition of the index of variability θ in the parameters h_* and h_{**}.

Classification of the approximations. The presence of three small parameters ε, h_*, and h_{**} generates a large diversity of asymptotics. We diminish the number of possible cases by using the following reasoning. For metals $\varepsilon \sim 10^{-5}$—10^{-3}, and there is no sense to keeping terms on the order of ε in comparison with unity. For thin shells ($h_* \lesssim 10^{-2}$) the components on the order of h_* should be discarded. For shells of medium thickness ($h_* \sim 10^{-1} \div 2 \cdot 10^{-1}$) taking account of corrections on the order of h_* may be of interest. Numerical computations show that two-dimensional theories sometimes work satisfactorily down to $h_{**} \sim 1/2$, and it is expedient to construct a two-dimensional theory in which corrections on the order of h_* and h_{**}^2 are taken into account.

In this connection, the following approximate theories are later examined:

The classical theory

$$(1 + [h_{**} + h_* + h_{**}^2 + h_{**}h_* + h_*^2 + \varepsilon + \ldots])$$

The fundamental refined theory

$$(1 + h_{**} + [h_* + h_{**}^2 + h_{**}h_* + h_*^2 + \varepsilon + \ldots])$$

The refined theory taking account of the geometric correction

$$(1 + h_{**} + h_* + [h_{**}^2 + h_{**}h_* + h_*^2 + \varepsilon + \ldots])$$

The theory taking account of transverse shear

$$(1 + h_{**} + h_* + h_{**}^2 + [h_{**}h_* + h_*^2 + \varepsilon + \ldots])$$

The orders of the terms conserved and discarded in comparison with one are indicated in the parentheses, while the terms are taken in the square brackets.

In separate parts of the shell l can be considerably less than R_0, and we have $h_* \sim h_{**}^2$ for $l \sim (hR_0)^{1/2}$. Hence, the theory taking account of the geometric correction can pretend to refine the classical theory only in those parts of a shell where $l \gg (hR_0)^{1/2}$.

Internal problem and boundary layer problem. Let us separate the functional I into the sum of two functionals:

$$I_1 = h \int_{V_1} U\varkappa \, d\omega \, d\zeta - \int_{S_0} P_i x^i (\xi^\alpha, \zeta) \, d\sigma \tag{3.4}$$

$$I_2 = h \int_{V_2} U\varkappa \, d\omega \, d\zeta - \int_{\Omega_2} \{P_i x^i (\xi^\alpha, \zeta) \varkappa\} \, d\omega$$

Let S_* denote a cylindrical surface separating the domains V_1 and V_2, and Γ_* its intersection with Ω_0; $S_* = \Gamma_* \times [-1/2, 1/2]$. Let the functions $x^i (\xi^\alpha, \zeta)$ be assigned in some manner on S_*

$$x^i (\xi^\alpha, \zeta) = x_*^i = r_*^i + hn_*^i \zeta + hy_*^i \tag{3.5}$$

Then problems to seek the stationary points of the functionals I_2 and I_1 in the set of functions $x^i (\xi^\alpha, \zeta)$ which satisfy the constraints (3.5) are defined. We call the first problem inner, and the second the boundary layer problem.

Certain estimates. We denote the number independent of h and the state of stress by m. Let d be the diameter of Ω_0 (the maximum distance between points of the curve Γ_0), $\Gamma_{0\beta\gamma}^\alpha$ and $\Gamma_{\beta\gamma}^\alpha$ are objects of connectedness on Ω_0 and Ω. We shall later understand l to be the lesser of the numbers d and the quantity l considered above. The following estimates hold

$$|a_{0\alpha\beta}| \leqslant m, \quad |a_0^{\alpha\beta}| \leqslant m, \quad |\Gamma_{0\beta\gamma}^\alpha| \leqslant m/d, \quad |\Gamma_{\beta\gamma}^\alpha| \leqslant m/l \tag{3.6}$$

$$h^2 C_{\alpha\beta} \equiv {}^1/_2 h^2 (c_{\alpha\beta} - c_{0\alpha\beta}) = h^2 b_{0(\alpha}^\lambda B_{\beta)\lambda} + O(\varepsilon_B{}^2 + h_* \varepsilon_A \varepsilon_B + h_*{}^2 \varepsilon_A{}^2) \tag{3.7}$$

Here $c_{\alpha\beta} = a^{\mu\nu} b_{\mu\alpha} b_{\nu\beta}$, symmetrization is denoted by the parentheses in the subscripts: $f_{(\alpha\beta)} \equiv {}^1/_2 (f_{\alpha\beta} + f_{\beta\alpha})$.

Let $\Delta_\alpha < 1$, $\Delta < 1$. Then it can be shown that

$$\Delta_\alpha \leqslant m\varepsilon, \quad \Delta \leqslant m\varepsilon, \quad \varepsilon_A \leqslant m\varepsilon, \quad \varepsilon_B \leqslant m\varepsilon \tag{3.8}$$

Inner problem. The condition of boundedness of the strain as $h \to 0$ imposes definite constraints on the external forces. The second term in the functional I_2 should be of the same order of smallness as the energy, i.e., $\mu\varepsilon^2 h \, |\Omega_0|$ ($|\Omega_0|$ is the area of Ω_0), or of higher order. Hence $P_i = O(h)$ as $h \to 0$. Let ε tend to zero. Since the external surface forces P_i are proportional to the strains, it is possible to write $P_i = O(\mu\varepsilon h)$. This estimate should contain a dimensionless small parameter and h should be replaced by h/l or h/R_0. It is natural to expect that the characteristic of the state of stress h/l figures in the estimate of the force. Thus, we take as fundamental assumption that

$$P_i = O(\mu\varepsilon h_{**}) \text{ on } \Omega_\pm.$$

Let us proceed to an asymptotic analysis of the functional I_2 in the case of a physically linear material

$$2U = \lambda (g_0^{\alpha\beta}\varepsilon_{\alpha\beta} + \varepsilon_{33})^2 + 2\mu g_0^{\alpha\nu} g_0^{\beta\delta}\varepsilon_{\alpha\beta}\varepsilon_{\nu\delta} + 2\mu\varepsilon_{33}^2 + 4\mu g_0^{\alpha\beta}\varepsilon_{\alpha 3}\varepsilon_{\beta 3} \tag{3.9}$$

Let us introduce the longitudinal and transverse elastic energies by means of the formulas

$$U_{\parallel} = \min_{\varepsilon_{\alpha 3}, \varepsilon_{33}} U, \quad U_{\perp} = U - U_{\parallel}$$

Simple calculations yield

$$U_{\parallel} = \mu [\sigma (g_0^{\alpha\beta}\varepsilon_{\alpha\beta})^2 + g_0^{\alpha\nu} g_0^{\beta\delta}\varepsilon_{\alpha\beta}\varepsilon_{\nu\delta}], \quad \sigma = \frac{\lambda}{\lambda + 2\mu}$$

$$U_{\perp} = {}^1/_2 (\lambda + 2\mu)(\varepsilon_{33} + \sigma g_0^{\alpha\beta}\varepsilon_{\alpha\beta})^2 + 2\mu g_0^{\alpha 3}\varepsilon_{\alpha 3}\varepsilon_{\beta 3}$$

Let us estimate the orders of y_α and y. To do this, it should be assumed that the functions $r^i(\xi^\alpha)$ are fixed and the first approximation for y_α, y should be found. The first approximation is sought from the condition of stationarity of the functional I_2 in which all small terms in the asymptotic sense are discarded. We proceed as follows. We first take the following approximate expressions for the strain

$$\varepsilon_{\alpha\beta} = A_{\alpha\beta} - hB_{\alpha\beta}\zeta, \quad 2\varepsilon_{\alpha 3} = y_{\alpha,\zeta} + hy_{,\alpha}, \quad \varepsilon_{33} = y_{,\zeta} \tag{3.10}$$

The functions y_α and y enter only in $U_\perp$. In the first approximation it is possible to set $\varkappa = 1$, $g_0^{\alpha\beta} = a_0^{\alpha\beta}$. Terms containing y_α, y have the form

$$h\int_{\Omega_2} U_\perp \, d\omega \, d\zeta = h\int_{\Omega_2} J \, d\omega \tag{3.11}$$

$$J = {}^1/_2 \langle (\lambda + 2\mu)(y_{,\zeta} + \sigma A_\alpha^\alpha - \sigma h B_\alpha^\alpha \zeta)^2 + \mu a_0^{\alpha\beta}(y_{\alpha,\zeta} + hy_{,\alpha})(\alpha \to \beta)\rangle$$

The symbol $(\alpha \to \beta)$ denotes the expression in the preceding parenthesis with the subscript α replaced by β. As yet the work of the external forces is discarded. The functional (3.11) "does not retain" the boundary conditions (3.5). Hence, the determination of y_α, y reduces to minimizing J under the constraints $\langle y_\alpha \rangle = \langle y \rangle = 0$. The minimum of J is zero and is achieved at the functions

$$y_\alpha = {}^1/_2 \sigma h A_{\beta,\alpha}^{\beta}(\zeta^2 - {}^1/_{12}) - {}^1/_6 \sigma h^2 B_{\beta,\alpha}^{\beta}\zeta(\zeta^2 - {}^1/_4) \tag{3.12}$$

$$y = -\sigma A_\alpha^\alpha \zeta + {}^1/_2 \sigma h B_\alpha^\alpha (\zeta^2 - {}^1/_{12})$$

Therefore, $y \sim \varepsilon_A + \varepsilon_B$, $y_\alpha \sim h_{**}(\varepsilon_A + \varepsilon_B)$. Now let us verify whether use of approximate expressions for the strain and discarding the work of the external forces would be allowable. We write the complete expression for the strains in the form (two bars in the subscripts indicated covariant differentiation on Ω, and the semicolon indicates covariant differentiation on Ω_0)

$$\varepsilon_{\alpha\beta} = A_{\alpha\beta} - hB_{\alpha\beta}\zeta + h^2 C_{\alpha\beta}\zeta^2 + h(y_{(\alpha\parallel\beta)} - b_{\alpha\beta}y) - h^2 b_{(\alpha}^{\mu} y_{\mu\parallel\beta)}\zeta + h^2 c_{\alpha\beta} y\zeta + {}^1/_2 h^2 (y_{,\alpha} + b_{\alpha\lambda}a^{\mu\lambda}y_\lambda)(\alpha \to 3) + {}^1/_2 h^2 a^{\mu\nu}(y_{\mu\parallel\alpha} - b_{\mu\alpha}y)(y_{\nu\parallel\beta} - b_{\nu\beta}y) \tag{3.13}$$

$$2\varepsilon_{\alpha 3} = y_{\alpha,\zeta} + hy_{,\alpha} + hb_{\alpha\lambda}a^{\mu\lambda}y_{\lambda} - hb_{\alpha\lambda}a^{\lambda\mu}y_{\mu,\zeta} + h(y_{,\alpha} + b_{\lambda\mu}a^{\mu\lambda}y_{\lambda})y_{,\zeta} + ha^{\mu\nu}(y_{\mu\|\alpha} - b_{\mu\alpha}\gamma)y_{\nu\|\zeta}$$

$$\varepsilon_{33} = y_{,\zeta} + {}^1/_2 y_{,\zeta}^2 + {}^1/_2 a^{\mu\nu}y_{\mu,\zeta}y_{\nu,\zeta}$$

It is seen from (3.13) that all the discarded terms in the expressions for $\varepsilon_{\alpha 3}$ and ε_{33} yield small corrections to the terms entering into (3.10). The member $h^2C_{\alpha\beta}\zeta^2$ can be omitted in the expression for $\varepsilon_{\alpha\beta}$ because of the estimate (3.7). The members containing y yield small terms compared to the terms in the functional J for the energy. Among the terms containing y_α the $hy_{(\alpha\|\beta)}$ has the lowest order of smallness. Let us examine the cross-terms they generate in the energy (the quadratic term $h^2y^2_{(\alpha\|\beta)}$ is small compared to the term $y^2_{\alpha,\zeta}$ in J according to the definition of the characteristic scale (3.3)).

$$h\langle g_0^{\alpha\nu}g_0^{\beta\delta}(A_{\alpha\beta} - hB_{\alpha\beta}\zeta)y_{\nu\|\delta}\varkappa\rangle \tag{3.14}$$

Let us note that for small h_* we have according to (2.2)

$$g_0^{\alpha\beta} = a_0^{\alpha\beta} + 2\xi b_0^{\alpha\beta} + O(h_*^2), \quad g_1^{\alpha\beta} \equiv (\varkappa)^{1/2}g_0^{\alpha\beta} = a_0^{\alpha\beta} + 2\xi b_1^{\alpha\beta} + O(h_*^2), \quad b_{1|}^{\alpha\beta} = b_0^{\alpha\beta} - {}^1/_2 H_0 a_0^{\alpha\beta} \tag{3.15}$$

Because of the constraint $\langle y_\alpha\rangle = 0$ we have $\langle A_{\alpha\beta}y_{\gamma\|\delta}\rangle = 0$. The principal cross term between y_α and $A_{\alpha\beta}$ in (3.14) has the form $h^2\langle a_0^{\alpha\nu}b_1^{\beta\delta}A_{\alpha\beta}y_{\gamma\|\delta}\zeta\rangle$ and therefore, is small compared to the cross term between $y_{\alpha,\zeta}$ and $y_{,\alpha}$. The principal cross term between y_α and $B_{\alpha\beta}$ in (3.14) $h^2\langle B_{\alpha\beta}y_{\gamma\|\delta}\zeta\rangle$ has the same order of smallness as the cross term between $y_{\alpha,\zeta}$ and $y_{,\alpha}$ in J. Hence, for the correct calculation of y_α it should be conserved.

The work of the forces Py ($P \equiv P_i n^i$) induces a contribution on the order of $\mu\varepsilon^2 h_{**}^2$ and is negligible for the determination of y in a first approximation. The work of the forces $P^\alpha y_\alpha$ ($P^\alpha \equiv P_i r_\beta^i a^{\alpha\beta}$) is on the order of $\mu\varepsilon^2 h_{**}^2$, and therefore, induces the same contribution as $\mu y^\alpha_{,\zeta}y_{\alpha,\zeta}$. It should hence be taken into account. Conservation of the terms mentioned does not change the orders of y_α and y and the formula for y in (3.12).

It follows from the estimates $y \sim \varepsilon_A + \varepsilon_B$, $y_\alpha \sim (\varepsilon_A + \varepsilon_B)h_{**}$ and (3,13) that $|\varepsilon_{\alpha\beta}| \leqslant m(\varepsilon_A + \varepsilon_B)$, $|\varepsilon_{\alpha 3}| \leqslant m(\varepsilon_A + \varepsilon_B)h_{**}$, $|\varepsilon_{33}| \leqslant m(\varepsilon_A + \varepsilon_B)$. Hence $\varepsilon \leqslant m(\varepsilon_A + \varepsilon_B)$ and, according to (3.8), the measures of smallness of the strain $\varepsilon_A + \varepsilon_B$ and ε are asymptotically equivalent.

Classical theory. The transverse part of the elastic energy is zero in a first approximation and the total energy agrees with the longitudinal energy calculated by strains $\varepsilon_{\alpha\beta} = A_{\alpha\beta} - hB_{\alpha\beta}\zeta$. Retaining the principal terms in $A_{\alpha\beta}$ and $hB_{\alpha\beta}$ and the principal cross term, we obtain

$$\Phi = h\langle U\rangle = h\langle U_\|\rangle = F(A) + {}^1/_{12}F(hB) + F(A, hB)$$

$$F(A) = \mu h[\sigma(A_\alpha^\alpha)^2 + A_{\alpha\beta}A^{\alpha\beta}]$$

$$F(hB) = \mu h[\sigma(hB_\alpha^\alpha)^2 + h^2B_{\alpha\beta}B^{\alpha\beta}]$$

$$F(A, hB) = {}^1/_6\mu h^3 b_1^{\gamma\delta}[\sigma(A_\alpha^\alpha B_{\gamma\delta} + A_{\gamma\delta}B_\alpha^\alpha) - 2A_\gamma^\alpha B_{\alpha\delta}]$$

The tensors $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are kinematically independent; for given $A_{\alpha\beta}$ the

tensor $B_{\alpha\beta}$ can have functional degrees of freedom. Additional degrees of freedom can be sought (for fixed $A_{\alpha\beta}$) from the conditions for the extremum of a functional with the energy density $^1/_{12}F(hB) + F(A, hB)$. The cross term $F(A, hB)$ hence plays the part of external effects for the additional degrees of freedom. However, in the majority of linear problems occurring in applications, assigning $A_{\alpha\beta}$ by virtue of the support conditions on the edge defines the tensor $B_{\alpha\beta}$ completely; by virtue of the inequality $F(A, hB) \leqslant mh_*(F(A) + F(hB))$ the cross energy is "subject" to the tension and bending energies and can be discarded. We consequently arrive at the formula $\Phi = F(A) + {}^1/_{12}F(hB)$.

Within the framework of the accuracy of the classical theory, the measures of middle surface strain can be overdefined because of the addition of small terms. For instance, the tensor $B_{\alpha\beta}$ can be replaced by the tensor [6]

$$\rho_{\alpha\beta} = B_{\alpha\beta} - b_{0(\alpha}^{\lambda}A_{\beta)\lambda} \tag{3.16}$$

This corresponds to adding components on the order of $h_*\varepsilon_A$ to hB and changes only the cross energy which is discarded in classical theory of shells. We shall later understand the energy density in classical theory of shells to be the quantity

$$\Phi = F(A) + {}^1/_{12}F(h\rho) \tag{3.17}$$

The work of the external forces on the front surfaces reduces in a first approximation to the work by $x^i = r^i(\xi^\alpha) + hn^i(\xi^\alpha)\zeta$. In an edge functional I_1 the energy is on the order of $\mu\varepsilon^2 |\Gamma_0| b$ ($|\Gamma_0|$ is the length of Γ_0) and is small compared to the energy of the inner part of the shell which is on the order of $\mu\varepsilon^2 |\Omega_0|$. Work of the forces in the edge functional reduces in a first approximation to work in $x^i = r^i(\xi^\alpha) + hn^i(\xi^\alpha)\zeta$ (this becomes clear after examination of the refined theories). Moreover, the integral over Ω_2 in the expression for the energy of the inner part of the shell can be replaced by an integral over Ω_0, this results in the addition of a small component on the order of $\mu\varepsilon^2 |\Gamma_0| b$. Thus the functional of the shell energy is given in a first approximation by the formula

$$I(r(\xi)) = \int_{\Omega_0} \Phi\, d\omega - L \tag{3.18}$$

$$L = \int_{\Omega_0} \left(\{P_i\} r^i + \frac{1}{2} h [P_i] n^i\right) d\omega + h \int_{\Gamma_0} (\langle P_i \rangle r^i + h \langle P_i \zeta \rangle n^i)\, ds$$

$$\{P_i\} = P_i|_{\zeta=1/2} + P_i|_{\zeta=-1/2}, \quad [P_i] = P_i|_{\zeta=1/2} - P_i|_{\zeta=-1/2}$$

Here Φ is the function (3.17) and ds is a length element on Γ_0.

4. **Surface energy in refined theories.** To calculate the terms in the energy density which are of order $\mu\varepsilon^2 h_*$, $\mu\varepsilon^2 h_{**}$, $\mu\varepsilon^2 h_{**}^2$, it is necessary to find the y_α which are of order εh_{**}, as has been established. Determination of y_α is connected with retention in energy terms of order $\mu\varepsilon^2 h_{**}^2$. Terms of such order enter into both the longitudinal and transverse energies. In order to simplify the calculations and not take account of terms in the longitudinal energy, we substitue in place of (3.1)

$$x^i = r^i(\xi^\alpha) + h\varphi^i(\xi^\alpha)\zeta + hz^i(\xi^\alpha, \zeta) \tag{4.1}$$

The functions r^i and φ^i are considered independent desired functions. The presence of three independent functions of ξ^α in addition permits the imposition of six constraints on the remainder z^i

$$\langle z^i \rangle = 0, \quad \langle z^i \zeta \rangle = 0 \tag{4.2}$$

The functions r^i and φ^i have the following meaning: $r^i = \langle x^i(\xi^\alpha, \zeta)\rangle$, $\varphi^i = 12h^{-1} \times \langle x^i(\xi^\alpha, \zeta)\,\zeta\rangle$. Formula (4.1) sets up a mutual one-to-one correspondence between all functions $x^i(\xi^\alpha, \zeta)$ and all triples $\{r^i, \varphi^i, z^i\}$ in which the z^i are subject to the constraintes (4.2). It follows from the preceding analysis that $\varphi_\alpha \equiv r^i_\alpha \varphi_i \sim \varepsilon h_{**}$, $\varphi - 1 \equiv n^i \varphi_i - 1 \sim \varepsilon$. Later we consider r^i, φ^i fixed and such that $\varphi_\alpha \sim \varepsilon h_{**}$, $\varphi - 1 \sim \varepsilon$ and we seek z^i.

Higher order derivatives of the functions desired must be estimated when constructing approximate theories possessing higher accuracy. In this connection, additional inequalities for the subsequent derivatives should be introduced in the definition of the characteristic scales. Within the framework of the substitution (4.1) this reduces to adding estimates of derivatives of the new degrees of freedom. We replace (3.3) by the inequalities

$$|A_{\alpha\beta,\gamma}| \leqslant \frac{\varepsilon_A}{l}, \quad \frac{1}{2}h|B_{\alpha\beta,\gamma}| \leqslant \frac{\varepsilon_B}{l} \tag{4.3}$$

$$|\varphi_\alpha| \leqslant \frac{(\varepsilon_A + \varepsilon_B)h}{l}, \quad |\varphi - 1| \leqslant \varepsilon_A + \varepsilon_B, \quad |\varphi_{,\alpha\beta}| \leqslant \frac{(\varepsilon_A + \varepsilon_B)h}{l^2}$$

$$|\varphi_{,\alpha}| \leqslant \frac{\varepsilon_A + \varepsilon_B}{l}, \quad \max_\zeta |z_{\alpha,\gamma}| \leqslant \frac{\overline{\Delta}}{l}, \quad \max_\zeta |z_{,\gamma}| \leqslant \frac{\overline{\Delta}}{l}$$

$$(z_\alpha \equiv r^i_\alpha z_i,\ z \equiv n^i z_i,\ \overline{\Delta} \equiv \max_{V_0}(|z_{1,\zeta}| + |z_{2,\zeta}| + |z_{,\zeta}|)$$

Since $z^i = y^i - r^i_\beta \varphi_\alpha a^{\alpha\beta}\zeta - (\varphi - 1)n^i\zeta$, $\overline{\Delta} \sim \varepsilon$. Let us write the strain components in terms of r^i, φ^i, z^i

$$\varepsilon_{\alpha\beta} = A_{\alpha\beta} - hB^{(\varphi)}_{\alpha\beta}\zeta + h^2 C^{(\varphi)}_{\alpha\beta}\zeta^2 + hr^i_{(\alpha} z_{i,\beta)} + h^2 \varphi^i_{(,\alpha} z_{i,\beta)}\zeta + {}^1/_2 h^2 z^i_{,\alpha} z_{i,\beta} \tag{4.4}$$

$$2\varepsilon_{\alpha 3} = z_{\alpha,\zeta} + \varphi_\alpha + h\varphi^i z_{i,\alpha} + {}^1/_2 h(\varphi_i \varphi^i)_{,\alpha}\zeta + h\varphi^i_{,\alpha} z_{i,\zeta}\zeta + hz^i_{,\alpha} z_{i,\zeta}$$

$$\varepsilon_{33} = z_{,\zeta} + {}^1/_2(\varphi_i\varphi^i - 1) + {}^1/_2 z^i_{,\zeta} z_{i,\zeta}$$

$$(B^{(\varphi)}_{\alpha\beta} \equiv -r^i_{(\alpha}\varphi_{i,\beta)} - b_{0\alpha\beta} = B_{\alpha\beta} - \varphi_{(\alpha\|\beta)} + (\varphi - 1)b_{\alpha\beta} = B_{\alpha\beta} + O(\varepsilon(h_* + h_{**})),\ C^{(\varphi)}_{\alpha\beta} \equiv {}^1/_2(\varphi^i_{,\alpha}\varphi_{i,\beta} - c_{0\alpha\beta}))$$

The estimates

$$|z^i_{,\alpha}| \leqslant m\overline{\Delta}(l^{-1} + R_0^{-1}),\ |\varphi^i_{,\alpha} - n^i_{,\alpha}| \leqslant m(\varepsilon_A + \varepsilon_B)(l^{-1} + R_0^{-1})$$

$$h^2|(\varphi^i_{,\alpha} - n^i_{,\alpha})z_{i,\beta}| \leqslant m(h_* + h_{**})^2 \varepsilon\overline{\Delta}$$

follow from (4.3). These estimates show that the last two members in the formula for $\varepsilon_{\alpha\beta}$ can be replaced by $h^2 n^i_{(,\alpha} z_{i,\beta)}\zeta$ to the accuracy of terms of the order of $(h_* + h_{**})^2\varepsilon^2$. The terms $hr^i_\alpha z_{i,\beta}$ and $h^2 n^i_{(,\alpha} z_{i,\beta)}\zeta$ yield a contribution on the order of $(h_*^2\varepsilon_A + h_*\varepsilon_B)(h_* + h_{**})\overline{\Delta}$ to the longitudinal energy because of the constraints

(4.2). Since there are cross terms between z^i and r^i, φ^i on the order of $h_{**}\varepsilon\bar{\Delta}$ in the transverse energy, the cross terms in the longitudinal energy can be discarded. Therefore, in seeking the z^i in a first approximation, their contribution to just the transverse energy must be taken into account. Simplifications because of discarding in the formula for

$\varepsilon_{\alpha\beta}\ \ h\varphi^i_{,\alpha}z_{i,\zeta}\zeta\ (|h\varphi^i_{,\alpha}z_{i,\zeta}\zeta| \leqslant m(h_* + h_{**})(\varepsilon_A + \varepsilon_B)\bar{\Delta} + mh_*\bar{\Delta})$ and $hz^i_{,\alpha}z_{,\zeta} \times (|hz^i_{,\alpha}z_{i,\zeta}| \leqslant m(h_* + h_{**})\bar{\Delta}^2)$

and the substitution

$$\varphi^i\varphi_i - 1 \text{ on } 2(\varphi - 1),\ \varphi^i z_{i,\alpha} \text{ on } z_{,\alpha}$$
$$(|\varphi_i\varphi^i - 1 - 2(\varphi - 1)| \leqslant m(\varepsilon_A + \varepsilon_B)^2,\quad |\varphi^i z_{i,\alpha} - n^i z_{i,\alpha}| \leqslant l^{-1}(\varepsilon_A + \varepsilon_B)^2 + mh_{**}\varepsilon\bar{\Delta})$$

are also possible in the transverse energy. The approximate expressions $g_{0\alpha\beta} = a_{0\alpha\beta}$, $g_0^{\alpha\beta} = a_0^{\alpha\beta}$ can be used for the metric tensor components since the values of z^i are sought in a first approximation. Taking account of the work of the external forces on z^i, we obtain the following functional to determine z^i

$$h\int_{\Omega_2}(J_\angle(z_\alpha, z) + J_\perp(z))\,d\omega \tag{4.5}$$

$$J_\angle(z_\alpha, z) = \frac{1}{2}\int_{-1/2}^{1/2}\mu a_0^{\alpha\beta}(z_{\alpha,\zeta} + \varphi_\alpha + h\varphi_{,\alpha}\zeta + hz_{,\alpha})(\alpha\to\beta)\,d\zeta - \{P^\alpha z_\alpha\}$$

$$J_\perp(z) = \frac{1}{2}\int_{-1/2}^{1/2}(\lambda + 2\mu)(z_{,\zeta} + \varphi - 1 + \sigma A^\alpha_\alpha - \sigma hB^\alpha_{\alpha\zeta})^2\,d\zeta - \{Pz\}$$

The functional (4.5) "does not maintain" the kinematic boundary conditions on S_*, hence, the evaluation of z_α and z reduces to minimization of the sum $J_\angle + J_\perp$ with respect to z_α and z at each point of the surface Ω_2, which are considered as arbitrary functions of ζ dependent on ξ^α as on the parameters and satisfying the constraints $\langle z_\alpha\rangle = \langle z_\alpha\zeta\rangle = \langle z\rangle = \langle z\zeta\rangle = 0$. To find z_α it is necessary to know z in a first approximation. The cross terms between z_α and z in the functional $J_\angle$ are small compared to the cross terms between z and $A_{\alpha\beta}$, $hB_{\alpha\beta}$ in $J_\perp$. Hence, the first approximation for z is determined from the problem of the minimum $J_\perp$. We obtain

$$z = {}^1/_2\sigma hB^\alpha_\alpha(\zeta^2 - {}^1/_{12}) - {}^5/_3\bar{\varphi}\zeta(\zeta^2 - {}^3/_{20}) + {}^1/_2(\lambda + 2\mu)^{-1}({}^5/_3[P]\zeta(\zeta^2 - {}^3/_{20}) + \{P\}(\zeta^2 - {}^1/_{12})) \tag{4.6}$$

$$\inf J_\perp = -{}^1/_{12}\sigma\{P\}hB^\alpha_\alpha + {}^5/_{12}(\lambda + 2\mu)\bar{\varphi}^2 + {}^1/_{12}[P]\bar{\varphi} \tag{4.7}$$

Here $\bar{\varphi} = \varphi - 1 + \sigma A^\alpha_\alpha$. The quadratic terms in the external force components are omitted in (4.7) since, as is easily verified, their variation yields a contribution of order εh^2_{**} to the equation as compared with one (in linear theory these terms are not variated and play the part of additive constants in the energy functional).

Let us assume that $\bar{\varphi} = O(\varepsilon h_{**})$ (this will be confirmed later). Then according to (4.6), in a first approximation

$$z = {}^1/_2\sigma h B_\alpha^\alpha\,(\zeta^2 - {}^1/_{12})$$

Substituting this expression in $J_\angle\,(z_\alpha, z)$ and minimizing with respect to z_α, we find

$$z_\alpha = -{}^1/_2 h\varphi_{,\alpha}\,(\zeta^2 - {}^1/_{12}) + {}^1/_6\sigma h^2 B^\beta_{\beta,\alpha}\zeta\,(\zeta^2 - {}^3/_{20}) - \tag{4.8}$$
$${}^5/_3\bar\varphi_\alpha\zeta\,(\zeta^2 - {}^3/_{20}) + {}^1/_2\mu^{-1}({}^5/_3\,[P_\alpha]\,\zeta\,(\zeta^2 - {}^3/_{20}) +$$
$$\{P_\alpha\}\,(\zeta^2 - {}^1/_{12})),\quad \bar\varphi_\alpha \equiv \varphi_\alpha - {}^1/_{60}\sigma h^2 B^\beta_{\beta,\alpha}$$
$$\inf J_\angle = h^{-1}\Phi_\angle + {}^1/_{12}\,[P^\alpha]\,\bar\varphi_\alpha + {}^1/_{12}\,\{P^\alpha\}\,h\varphi_{,\alpha} +$$
$${}^1/_{120}\sigma\,[P^\alpha]\,h^2 B^\beta_{\beta,\alpha},\quad \Phi_\angle = h\,\langle U_\angle\rangle = {}^5/_{12}\mu h\bar\varphi_\alpha\bar\varphi^\alpha$$

Quadratic terms in the external forces are discarded in (4.8).

To construct the energy functional of the inner part of the shell there remains to evaluate the surface density of the longitudinal energy $\Phi_\parallel = h\,\langle U_\parallel\varkappa\rangle$. Here the approximate expression $\varepsilon_{\alpha\beta} = A_{\alpha\beta} - hB^{(\varphi)}_{\alpha\beta}\zeta + h^2 C^{(\varphi)}_{\alpha\beta}\zeta^2$ can be used for the strain components $\varepsilon_{\alpha\beta}$ (let us note that $C^{(\varphi)}_{\alpha\beta} = C_{\alpha\beta} + O\,(h_*\,(h_* + h_{**})\,\varepsilon)$). We introduce the nonsymmetric tensor $\varepsilon^\alpha_{1\beta} = g_1^{\alpha\gamma}\varepsilon_{\gamma\beta}$, where $g_1^{\alpha\gamma}$ are determined by (3.15). The energy density can be given the form

$$U_\parallel\varkappa = \mu\,[\sigma\,(\varepsilon^\alpha_{1\alpha})^2 + \varepsilon^\alpha_{1\beta}\varepsilon^\beta_{1\alpha}]$$

Re-expanding $\varepsilon^\alpha_{1\beta}$ in the first Legendre polynomials, we obtain by using (3.15)

$$\varepsilon^\alpha_{1\beta} = (A^\alpha_\beta - {}^1/_6 h^2 b_1^{\alpha\gamma}B^{(\varphi)}_{\gamma\beta} + {}^1/_{12}h^2 C^\alpha_\beta) -$$
$$h\,(B^{(\varphi)\alpha}_{\ \ \beta} - 2b_1^{\alpha\gamma}A_{\gamma\beta})\,\zeta + h^2(\ldots)\,(\zeta^2 - {}^1/_{12}) + \ldots$$

We omit the superscript in the $\varepsilon^\alpha_{1\beta}$ by using the metric $a_{0\alpha\beta}$. Then

$$U_\parallel\varkappa = \mu\,[\sigma\,(a_0^{\alpha\beta}\varepsilon_{1\alpha\beta})^2 + a_0^{\alpha\gamma}a_0^{\beta\nu}\varepsilon_{1\alpha\beta}\varepsilon_{1\gamma\delta}] \tag{4.9}$$

The antisymmetric part of $\varepsilon_{1\alpha\beta}$ yields a contribution on the order of $h_*^2\varepsilon^2$ to $U_\parallel\varkappa$ and cannot be taken into account. Hence, to evaluate $\Phi_\parallel$ it is necessary to substitute the tensor $\varepsilon_{1(\alpha\beta)} = A_{1\alpha\beta} - hB_{1\alpha\beta}\zeta$ in place of $\varepsilon_{1\alpha\beta}$ in (4.9), where

$$A_{1\alpha\beta} = A_{\alpha\beta} - {}^1/_6 h^2 b_{1(\alpha}^{\ \ \lambda}B^{(\varphi)}_{\beta)\lambda} + {}^1/_{12}h^2 C_{\alpha\beta},\quad B_{1\alpha\beta} = B^{(\varphi)}_{\alpha\beta} - 2b_{1(\alpha}^{\ \ \lambda}A_{\beta)\lambda}$$

We obtain relationship for $\Phi_\parallel$ which agrees with the classical in form: $\Phi_\parallel = F\,(A_1) + {}^1/_{12}F\,(hB_1)$. The difference is in the expressions for the deformation and the bending measures. We find φ. The derivatives of φ enter the functional linearly, hence, by varying with respect to φ, we obtain a linear algebraic equation for φ

$${}^5/_6\,(\lambda + 2\mu)(\varphi - 1 + \sigma A^\alpha_\alpha) + {}^1/_{12}\,[P] - {}^1/_{12}\,\{P^\alpha\}_{;\alpha} +$$
$$h^{-1}\,b_{\alpha\beta}\partial\Phi_\parallel / \partial B_{1\alpha\beta} = 0$$

It is hence seen that, as had been assumed above, $\varphi = 1 - \sigma A^\alpha_\alpha$ in a first approximation. The error of this formula is $O\,(\varepsilon\,(h_* + h_{**}))$. As a measure of the bending, we take the tensor

$$\rho_{\alpha\beta} = B_{\alpha\beta} - \varphi_{(\alpha;\beta)} - b_{0(\alpha}^{\ \ \lambda}A_{\beta)\lambda}$$

It agrees with the tensor (3.16) if the transverse sheer φ_α is not taken into account.

The tensors $A_{1\alpha\beta}$ and $B_{1\alpha\beta}$ in terms of $A_{\alpha\beta}$, $\rho_{\alpha\beta}$ with the expression for φ taken into account, have the form ($b'_{\alpha\beta} = b_{0\alpha\beta} - H_0 a_{0\alpha\beta}$ is the deviator of the second quadratic form of Ω_0).

$$A_{1\alpha\beta} = A_{\alpha\beta} - {}^1/_{12}h^2 b'^{\lambda}_{(\alpha}\rho_{\beta)\lambda}, \quad B_{1\alpha\beta} = \rho_{\alpha\beta} - b'^{\lambda}_{(\alpha}A_{\beta)\lambda} - \sigma b_{0\alpha\beta}A^{\lambda}_{\lambda}$$

The expression for the energy still remains awkward. Especially "unacceptable" is the gradient B^{β}_{β}, which enters into the transverse shear energy in terms of $\overline{\varphi}_\alpha = \varphi_\alpha - {}^1/_{60}h^2 B^{\beta}_{\beta,\alpha}$. Since φ_α are independent functions, it is natural to get rid of the gradient B^{β}_{β} by going over to new desired functions $\overline{\varphi}_\alpha$. Here it is also necessary to make the substituion $r_i \to \bar{r}_i = r_i - {}^1/_{60}\sigma h^2 B^{\beta}_{\beta} n_i$ in order that the formula for $\rho_{\alpha\beta}$ does not change (this formula was used in [29] for the linear theory of plates). Let $\gamma_{\alpha\beta}$ denote the tensor $A_{\alpha\beta}$ constructed with respect to $\bar{r}_i$: $\gamma_{\alpha\beta} = {}^1/_2(\bar{r}_{,\alpha}\bar{r}_{i,\beta} - a_{0\alpha\beta})$, and $\bar{n}_i$ the vector determined from the relationships $\bar{n}_i\bar{r}^{i}_{,\alpha} = 0$, $\bar{n}_i\bar{n}^i = 1$. Then

$$A_{1\alpha\beta} = \gamma_{\alpha\beta} - {}^1/_{12}h^2 b'^{\lambda}_{(\alpha}\rho_{\beta)\lambda} - {}^1/_{60}\sigma h^2 b_{0\alpha\beta}\rho^{\mu}_{\mu} \tag{4.10}$$
$$B_{1\alpha\beta} = \rho_{\alpha\beta} - b'^{\lambda}_{(\alpha}\gamma_{\beta)\lambda} - \sigma b_{0\alpha\beta}\gamma^{\lambda}_{\lambda}, \quad \rho_{\alpha\beta} = \bar{n}_i\bar{r}^{i}_{,\alpha\beta} - b_{0\alpha\beta} - \overline{\varphi}_{(\alpha;\,\beta)} - b_{0(\alpha}^{\lambda}\gamma_{\beta)\lambda}$$

After substituting (4.10) into $\Phi_{\|}$ and discarding terms on the order of h^2_* compared with the principal terms, the surface elastic energy density $\Phi = h\langle U\varkappa\rangle = h\langle U_{\|}\varkappa\rangle + h\langle U_{\perp}\varkappa\rangle$ is written in the form

$$\Phi = F(\gamma) + {}^1/_{12}F(h\rho) + F(\gamma,\ h\rho) + {}^5/_{12}\mu h\overline{\varphi}_\alpha\overline{\varphi}^\alpha \tag{4.11}$$
$$F(\gamma,\ h\rho) = -{}^1/_3\mu h^3(\gamma_{\alpha\beta}b'^{\alpha}_{\lambda}\rho^{\lambda\beta} + \sigma\gamma^{\lambda}_{\lambda}b_{0\alpha\beta}\rho^{\alpha\beta} + {}^3/_5\sigma\gamma^{\alpha\beta}b_{0\alpha\beta}\rho^{\lambda}_{\lambda} + \sigma({}^6/_5\sigma - 1)\gamma^{\alpha}_{\alpha}H_0\rho^{\beta}_{\beta})$$

Here $F(\gamma)$ and ${}^1/_{12}F(h\rho)$ is the tension and bending energies according to classical theory, $F(\gamma, h\rho)$ is the cross energy, and the last term in (4.11) is the shear energy.

The work of the external forces is given by

$$L_2 = \int_{\Omega_2}\left((\{P_i\} - hH_0[P_i])\bar{r}^i + \frac{1}{2}[P_i]h\bar{n}^i - \Theta\right)d\omega \tag{4.12}$$
$$\Theta = {}^1/_2\sigma h([P] + \underline{{}^1/_6 h\{P^\alpha\}_{;\alpha}})\gamma^{\beta}_{\beta} -$$
$${}^1/_{10}\sigma h^2(\{P\} + \underline{{}^1/_{12}h[P^\alpha]_{;\alpha}})\rho^{\beta}_{\beta} - {}^5/_{12}[P^\alpha]h\overline{\varphi}_\alpha$$

Formulas (4.11) and (4.12) determine the energy functional of the inner part of the shell.

5. Boundary layer problem. The boundary layer problem is a problem about the deformation of a closed elastic rod on a part of whose side surface (S_*) the positions $x^i = x^i_*(\xi^\alpha, \zeta)$ are given for the particles, and surface forces on another part (S_0), while the side surfaces $\zeta = \pm {}^1/_2$ are load-free. We consider the position of the particles on S_* to be matched with the internal expansion. This means that $x^i_* = r^i_* + h\varphi^i_*\zeta + hz^i_*$ (the asterisks denote values of the quantities on S_*), and the first approximation for z^i_* is found from (4.6), (4.8).

Since $U \sim \mu\varepsilon^2$, the volume integral in I_1 is on the order of $\mu\varepsilon^2 |\Gamma_0| h^2$ (for $b \sim h$). Its influence is felt in the boundary conditions in terms on the order of h_{**} as compared with one. Hence, I_1 must be found on the basis of the refined theory in a first approximation, and with corrections on the order of h_{**} compared to unity in the theory taking shear into account. Still another characteristic parameter occurs in the boundary layer problem, the surface curvature $k_{(n)}$ of the contour Γ_* (or Γ_0): $k_{(n)} = \tau_{0i} dv_0^i / ds$ ($\boldsymbol{\tau}_0$ is the tangent vector to Γ_*, $\vec{\nu}_0 = \vec{\tau}_0 \times \vec{n}_0$, and s is the arclength on Γ_* which increases if the surface Ω_2 remains on the left during traversal to Γ_*). We further consider that $k_{(n)} \sim l^{-1}$ and $a_{011} = a_{022} = 1, a_{012} = 0$ on Γ_*.

Energy of the free edge. First let $P_i = 0$ on S_0. To the accuracy of terms on the order of h_* and h_{**}, the metric tensor components in the coefficients of the quadratic form U can be replaced by the Kronecker symbols. Let us extract the longitudinal energy, the transverse energy, and the shear energy in U (the Greek subscripts in Sect. 5 run through the values 1, 3, E is the Young's modulus, and ν is the Poisson's ratio)

$$U = U_\parallel + U_\angle + U_\perp, \quad U_\parallel \equiv \min_{\varepsilon_{\alpha\beta}, \varepsilon_{\alpha 2}} U = {}^1/_2 E \varepsilon_{22}^2 \tag{5.1}$$

$$U_\angle \equiv \min_{\varepsilon_{\alpha\beta}} (U - U_\parallel) = 2\mu (\varepsilon_{21}^2 + \varepsilon_{23}^2)$$

$$U_\perp \equiv U - U_\parallel - U_\angle = \mu [(\varepsilon_{11} + \nu\varepsilon_{22})^2 + (\varepsilon_{33} + \nu\varepsilon_{22})^2 + 2\varepsilon_{13}^2] + {}^1/_2\lambda (\varepsilon_{11} + \varepsilon_{33} + 2\nu\varepsilon_{22})^2$$

It is convenient to introduce a new comoving coordinate system η, s, ζ' in the boundary layer problem, which would be given by the formulas

$$x^i = r_0^i(s) + h\nu_0^i(s)\eta + hn_0^i(s)\zeta'$$

in the initial state, where $x^i = r_0{}^i(s)$ is the equation of Γ_* in the initial state.

The rod cross section is curvilinear in the η, ζ' coordinates (Fig. 1).

The transition matrix from the coordinates $(\xi^1 - b)/h$, ζ to the coordinates η, ζ' differs from one by a quantity on the order of h_*, hence the energy in the new coordinates has the form (5.1) with the accuracy taken. The coordinates ζ and ζ' coincide for $\eta = 0$. The prime on the ζ' is further omitted since the coordinate ζ is not used in the boundary layer.

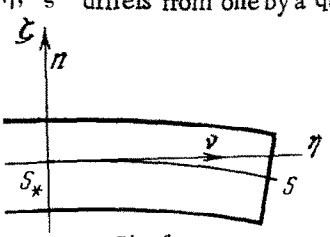

Fig. 1

Let us make a change in the required functions

$$x^i = r^i(s) + h\nu^i(s)\eta + hn^i(s)\zeta + hy^i(\eta, \zeta, s) \tag{5.2}$$

The vectors ν^i and n^i are located in the deformed state relative to the image of Γ_*, exactly as ν_0^i and n_0^i relative to Γ_*. Substituting (5.2) into the expression for the strain tensor components yields

$$\varepsilon_{11} = \underline{y_{1|1}} + {}^1/_2 y^i_{|1} y_{i|1}, \quad 2\varepsilon_{13} = \underline{y_{1|3}} + \underline{y_{3|1}} \tag{5.3}$$

$$\varepsilon_{33} = \underline{y_{3|3}} + {}^1/_2 y^i_{|3} y_{i|3}$$

$$2\varepsilon_{21} = \underline{y_{2|1}} - \underline{h\omega_*\zeta} + \underline{h\nu^i y_{i,s}} + \underline{hn^i_{,s} y_{i|1}\zeta} + hn^i_{,s} y_{i|1} + hy^i_{,s} y_{i|1}$$

$$2\varepsilon_{23} = y_{2|3} + h\omega_*\eta + hn^i y_{i,s} + h\nu^i_{,s} y_{i|3}\eta + hn^i_{,s} y_{i|3} + hy^i_{,s} y_{i|3}$$

$$\varepsilon_{22} = A^*_{22} - hB^*_{22}\zeta + h(\tau_i\nu^i_{,s} - \tau_{0i}\nu^i_{0,s})\eta + h\tau^i y_{i,s} + {}^1/_2h^2(n^i_{,s}n_{i,s} - n^i_{0,s}n_{0i,s})\zeta^2 + {}^1/_2h^2(\nu^i_{,s}\nu_{i,s} - \nu^i_{0,s}\nu_{0i,s})\eta^2 + h^2(n^i_{,s}\nu_{i,s} - n^i_{0,s}\nu_{0i,s})\eta\zeta + {}^1/_2h^2 y^i_{,s} y_{i,s} + h^2 n^i_{,s} y_{i,s}\zeta + h^2\nu^i_{,s} y_{i,s}\eta$$

The bar before the subscripts 1 and 3 here denotes the derivative with respect to η and ζ, and the comma before the s, the derivative with respect to s.

$$y_1 = \nu^i y_i, \quad y_2 = \tau^i y_i, \quad y_3 = n^i y_i, \quad \tau^i \equiv dr^i_* / ds$$
$$\omega_* = n_i\nu^i_{,s} - b_{012}, \quad A^*_{22} = {}^1/_2(\tau_i\tau^i - 1), \quad B^*_{22} = n_i\tau^i_{,s} - n_{0i}\tau^i_{0,s}$$

The quantities $h\omega_*$ and hB^*_{22} differ by terms on the order of $h_*\varepsilon_A$ from the values of hB_{12} and hB_{22} on Γ_*. Let us keep the terms underlined by the solid line in (5.3). Then the boundary layer problem dissociates into two independent problems, the plane problem of the minimum of a functional ($\langle\langle\cdot\rangle\rangle$ is the integral with respect to η, ζ over the rod cross section)

$$\langle\langle U_\perp\rangle\rangle = \langle\langle \mu[(y_{1|1} + \nu A^*_{22} - \nu hB^*_{22}\zeta)^2 + (y_{3|3} + \nu A^*_{22} - \nu hB^*_{22}\zeta)^2 + {}^1/_2(y_{1|3} + y_{3|1})^2] + {}^1/_2\lambda(y_{1|1} + y_{3|3} + 2\nu A^*_{22} - 2\nu hB^*_{22}\zeta)^2\rangle\rangle \tag{5.4}$$

in a set of functions y_1, y_3 satisfying the conditions

$$y_1 = 0, \quad y_3 = -\sigma A^{*\alpha}_{\alpha} + {}^1/_2\sigma hB^{*\alpha}_{\alpha}(\zeta^2 - {}^1/_{12})$$

for $\eta = 0$, and the antiplane problem of the minimum of the functional

$$\langle\langle U_\angle\rangle\rangle = {}^1/_2\mu\langle\langle (y_{2|1} - h\omega_*\zeta)^2 + (y_{2|3} + h\omega_*\eta)^2\rangle\rangle \tag{5.5}$$

in a set of functions y_2 which vanish for $\eta = 0$.

It is seen from (5.4) and (5.5) that y_1, y_2 and y_3 will be on the order of ε. Hence, all the terms discarded in (5.3) are less than the rest (the third member in the formula for ε_{22} differs slightly from $h\,\partial A_{22}/\partial\xi^1$ and, therefore, is of the order of $h_{**}\varepsilon_A$).

The integral over the cross-section $\langle\langle\cdot\rangle\rangle$ can be replaced by an integral over the rectangle $0 \leqslant \eta \leqslant b_* = b/h$, $|\zeta| \leqslant {}^1/_2$. This induces an error on the order of h_*.

Plane problem. Let us pass from y_1, y_3 to the new desired functions z_1, z_3

$$y_1 = -\nu A^*_{22}\eta + \nu hB^*_{22}\eta\zeta + z_1$$
$$y_3 = -\nu A^*_{22}\zeta + {}^1/_2\nu hB^*_{22}(\zeta^2 - {}^1/_{12}) - {}^1/_2\nu hB^*_{22}\eta^2 + z_3 \tag{5.6}$$

Then the functional of the plane problem takes the form

$$\langle\langle U_\perp\rangle\rangle = \langle\langle \mu[z_{1|1}{}^2 + {}^1/_2(z_{1|3} + z_{3|1})^2 + z_{3|3}{}^2] + {}^1/_2\lambda(z_{1|1} + z_{3|3})^2\rangle\rangle$$

For $\eta = 0$ the functions z_1, z_3 satisfy the conditions

$$z_1 = 0, \quad z_3 = (-\sigma A^{*\alpha}_{\alpha} + \nu A^*_{22})\zeta + {}^1/_2\, h\,(\sigma B^{*\alpha}_{\alpha} - \nu B^*_{22}) \times (\zeta^2 - {}^1/_{12})$$

Since to the accuracy assumed

$$A^*_{11} = \langle y_1\rangle_{|1}|_{\eta=0} = -\nu A^*_{22} + \langle z_{1|1}\rangle|_{\eta=0}$$

$$B^*_{11} = -\langle y_{3|11}\rangle|_{\eta=0} = -\nu B^*_{22} - \langle z_{3|11}\rangle|_{\eta=0}$$

and $\nu - \sigma = -\nu\sigma$, the boundary condition for z_3 is homogeneous

$$z_3 = -\sigma\langle z_{1|1}\rangle\zeta - {}^1/_2 h\sigma\,\langle z_{3|11}\rangle, \quad \eta = 0$$

Hence, $\inf\langle\!\langle U_\perp\rangle\!\rangle = 0$ and is reached for $z_1 = z_3 = 0$

Antiplane problem. The minimal value of $\langle\!\langle U_\angle\rangle\!\rangle$ is proportional to $b_*\,(h\omega_*)^2$. We denote the proportionality factor by D

$$\inf_{y_2}\langle\!\langle U_\angle\rangle\!\rangle = {}^1/_{12}\, D b_* h^2 \omega_*^2$$

Calculations yield

$$D = 2\mu\left[1 - \frac{h}{b}\,\frac{96}{\pi^5}\sum_{s=1}^{\infty}\frac{\tanh(2s-1)\,\pi b/h}{(2s-1)^5}\right]$$

For $h \leqslant b$ we can put approximately $\tanh(2s-1)\pi b/h = 1$ and $D \approx 2\mu$ $[1 - 0{,}315h/b]$.

The edge energy is comprised of the longitudinal energy $\langle\!\langle U_\parallel\rangle\!\rangle$ and the shear energy $\langle\!\langle U_\angle\rangle\!\rangle$. In a first approximation we have

$$\inf I_1 = \int_{\Gamma_*} \Psi\, ds, \quad \Psi = \frac{hb}{2}\left[E\left(A^{*2}_{22} + \frac{h^2}{12}B^{*2}_{22}\right) + \frac{h^2 D}{6}\,\omega_*{}^2\right] \tag{5.7}$$

The next approximation for the energy of a free edge. The taking account of corrections of the order h_{**} is associated with keeping terms emphasized by primes and a dot in (5.3). If $kk_{(n)} \sim h_*$, then the terms stressed by the dot can be omitted. It is easy to verify that $\inf\langle\!\langle U_\perp\rangle\!\rangle \sim \mu\varepsilon^2 h^2_{**}$ and does not yield a contribution to the edge energy. To evaluate $\inf\langle\!\langle U_\angle\rangle\!\rangle$ it is necessary to substitute the values of y_1 and y_3 found in a first approximation in the members of the expressions for ε_{21} and ε_{23} stressed by primes. The edge energy is easily found, however the answer is too awkward to set down here.

Functional of the energy of a loaded edge. The longitudinal components of the stress tensor $p^{\alpha\beta}$ in the internal part of the shell are on the order of $\mu\varepsilon$, while the transverse tensor $p^{\alpha 3}$ is on the order of $\mu\varepsilon^2 h_{**}$ In this connection, the external forces on the edge are naturally subjected to the condition

$$P_i\nu^i = O(\mu\varepsilon), \quad P_i\tau^i = O(\mu\varepsilon), \quad P_i n^i = O(\mu\varepsilon h_{**})$$

However, it turns out that the self-equilibrated part of the load yields a contribution to the edge energy for such "large" external forces. In fact, $\langle P_i\tau^i y_2\rangle \sim \mu\varepsilon^2$, i.e., is on the order of the shear energy $\langle\!\langle U_\angle\rangle\!\rangle$. The work $P_i\tau^i$ on y_2 will therefore enter into the functional of the antiplane problem. By solving the antiplane

problem, it can be seen that the self-equilibrated part of the load, equal to $P_i\tau^i - \langle P_i\tau^i\rangle - \zeta\langle P_i\tau^i\zeta\rangle / \langle\zeta^2\rangle$, performs work on $h\omega_*$ and, hence, is essential for the calculation of corrections of order h_{**} to the solution. Therefore, if only the total force and moment on the edge are known, then the maximum allowable accuracy is given by classical shell theory and the construction of refined theories is generally meaningless. The situation is changed if the external forces satisfy an additional constraint: (the part of $P_i\tau^i$ odd in ζ) $= O(\mu\varepsilon h_{**})$. The part of $P_i\tau^i$ even in ζ can be discarded in the antiplane problem since it does not work on $h\omega_*$ and yields only quadratic components in the external forces. The quantity $\langle$(the part of $P_i\tau^i$ odd in ζ)$\cdot y_2\rangle$ is of the order of $\mu\varepsilon^2 h_{**}$ and negligible compared to $\langle\langle U_\angle\rangle\rangle$. Hence, the taking account of the external forces does not change the solution of the antiplane problem. Work of the external forces on y_3 can be discarded since $\langle P_i n^i y_3\rangle \sim \mu\varepsilon^2 h_{**}$. There remains the work on y_1: $\langle P_i\nu^i y_1\rangle$. It is of the order of $\mu\varepsilon^2$ and will enter the functional of the plane problem. After the substitution of (5.6), it will become

$$\langle\langle\mu\,[z_{1|1}^2 + {}^1/_2\,(z_{1|3} + z_{3|1})^2 + z_{3|3}^2] + {}^1/_2\lambda\,(z_{1|1} + z_{3|3})^2\rangle\rangle - \langle P_i\nu^i z_1\rangle\,|_{\eta=b_*} - \langle P_i\nu^i\,(-\nu A_{22}^*\eta + \nu h B_{22}^*\eta\zeta)\rangle\,|_{\eta=b_*}$$

The extremals depend on $P_i\nu^i$. Discarding the terms dependent on $P_i\nu^i$, we obtain $\nu b_*\,(\langle P_i\nu^i\rangle A_{22}^* - \langle P_i\nu^i\zeta\rangle h B_{22}^*)$ for the minimum value of the functional of the plane problem. Consequently, the edge functional is given by the formula

$$I_1 = \int_{\Gamma_*} \Psi\,ds - L_1$$

$$L_1 = h\int_{\Gamma_*} (\langle P_i\rangle r^i + b\langle P_i\rangle\nu^i + h\langle P_i\zeta\rangle n^i - \nu b\,(\langle P_i\nu^i\rangle A_{22}^* - \langle P_i\nu^i\zeta\rangle hB_{22}^*))\,ds$$

(Ψ is the function in (5.7))

Construction of the edge functional with corrections on the order of h_{**} taken into account, and therefore, of a theory taking account of transverse shear is impossible if the self-equilibrated part of the load is unknown. Corrections of such an order are caused, for instance, by the self-equilibrated part of the force $P_i\tau^i$ in the antiplane problem. In this connection, a theory with shear taken into account is not universal and its application is justified only for special classes of problems (closed shells, shells with a free edge).

The above exposition permits the expectation that the following analog of the Saint Venant principle in shell statics is valid: in order for a load which is self-equilibrated in each transverse fiber of the edge to cause an internal state of stress representing $o(\mu\varepsilon h_{**})$, it is necessary and sufficient that $P_i\nu^i = O(\mu\varepsilon)$, $P_i n^i = O(\mu\varepsilon h_{**})$ and (the part of $P_i\tau^i$ odd in ζ) $= o(\mu\varepsilon)$.

The necessity of the latter condition is verified by the exact solution of the problem on the strain of a semi-infinite plate which is self-equilibrated by a load at the edge, which has been constructed in [30].

6. Two-dimensional refined theories of physically linear shells. Let us summarize the results. The function of a two-dimensional shell theory is given by the formula

$$I = \int_{\Omega_2} \Phi \, d\omega + \int_{\Gamma_*} \Psi \, ds - L, \quad L = L_1 + L_2 \tag{6.1}$$

Varying (6.1), within the framework of the accuracy of theories taking shear into account, we obtain a system of equations

$$t^{\alpha\beta}{}_{;\beta} - q^\beta b_{0\beta}^\alpha + \{P^\alpha\} = {}^1/_2 h\,[P^\beta](b_{0\beta}^\alpha + 2H_0\delta_\beta^\alpha) \tag{6.2}$$

$$q^\alpha{}_{;\alpha} + t^{\alpha\beta}b_{\alpha\beta} + \{P\} + {}^1/_2 h\,[P^\alpha]_{;\alpha} = hH_0\,[P] \tag{6.3}$$

$$t^{\alpha\beta} = n^{\alpha\beta} + {}^1/_2\,(b_{0\sigma}^\alpha m^{\beta\sigma} - b_{0\sigma}^\beta m^{\alpha\sigma}), \quad q^\alpha + m^{\alpha\beta}{}_{;\beta} = 0 \tag{6.4}$$

$$n^{\alpha\beta} = \frac{\partial(\Phi + \Theta)}{\partial\gamma_{\alpha\beta}}, \quad m^{\alpha\beta} = \frac{\partial(\Phi + \Theta)}{\partial\rho_{\alpha\beta}} \tag{6.5}$$

The classical terms are written in the left sides of (6.2) and (6.3), and corrections of order h_* in the right sides. The system (6.2)—(6.5) is closed in the first two refined theories. Two more equations for the function $\overline{\varphi}_\alpha$ are added to (6.2)—(6.5) in theories taking shear into account

$$q^\alpha = \frac{\partial(\Phi + \Theta)}{\partial\overline{\varphi}_\alpha} = \frac{5}{6}\mu h\overline{\varphi}^\alpha - \frac{5}{12}h\,[P^\alpha] \tag{6.6}$$

In the classical and the funadmental refined theories $\Phi = F(\gamma) + {}^1/_{12}F(h\rho)$ (if the cross energy is inessential), $\Theta = 0$. Hence

$$n^{\alpha\beta} = 2\mu h\,(\sigma\gamma_\lambda^\lambda a_0^{\alpha\beta} + \gamma^{\alpha\beta}), \quad m^{\alpha\beta} = {}^1/_6\mu h^3\,(\sigma\rho_\lambda^\lambda a_0^{\alpha\beta} + \rho^{\alpha\beta}) \tag{6.7}$$

In the theory taking account of the geometric correction $\Phi = F(\gamma) + {}^1/_{12}F(h\rho) + F(\gamma, h\rho)$, where $F(\gamma, h\rho)$ is given by (4.11), and small terms stressed by a dot should be discarded in the expression (4.12) for Θ. Therefore

$$\begin{aligned} n^{\alpha\beta} &= n_1^{\alpha\beta} - {}^1/_3\mu h^3\,[\rho^{\lambda(\alpha}b_\lambda^{\beta)} + \sigma\,(b_0^{\mu\nu}\rho_{\mu\nu} + ({}^6/_5\sigma - 1)\,H_0\rho_\mu^\mu)\,a_0^{\alpha\beta} + \\ &\quad {}^3/_5\sigma\rho_\lambda^\lambda b_0^{\alpha\beta}] + {}^1/_2\sigma h\,[P]\,a_0^{\alpha\beta} \\ m^{\alpha\beta} &= m_1^{\alpha\beta} - {}^1/_3\mu h^3\,[\gamma^{\lambda(\alpha}b_\lambda^{\beta)} + \sigma\,({}^3/_5 b_0^{\mu\nu}\gamma_{\mu\nu} + ({}^6/_5\sigma - 1)\,H_0\gamma_\mu^\mu)\,a_0^{\alpha\beta} + \\ &\quad \sigma\gamma_\lambda^\lambda b_0^{\alpha\beta}] - {}^1/_{10}\sigma h^2\,\{P\}\,a_0^{\alpha\beta} \end{aligned} \tag{6.8}$$

Here $n_1^{\alpha\beta}$, $m_1^{\alpha\beta}$ are tensors in (6.7).

In the theory taking account of shear Φ and Θ are given by (4.11) and (4.12). According to (6.5)

$$n^{\alpha\beta} = n_2^{\alpha\beta} + {}^1/_{12}\,\sigma h^2\,\{P^\mu\}_{;\mu}a_0^{\alpha\beta}, \quad m^{\alpha\beta} = m_2^{\alpha\beta} - {}^1/_{120}\sigma h^3[P^\mu]_{;\mu}\,a_0^{\alpha\beta}$$

where $n_2^{\alpha\beta}$ and $m_2^{\alpha\beta}$ are the tensors (6.8).

To write the boundary conditions we introduce the curvature of the contour Γ_* by the relationships [16]

$$\frac{dv_0^i}{ds} = k_{(n)}\tau_0^i - k_{(\tau)}n_0^i, \quad \frac{d\tau_0^i}{ds} = k_{(\nu)}n_0^i - k_{(n)}\nu_0^i$$
$$\frac{dn_0^i}{ds} = k_{(\tau)}\nu_0^i - k_{(\nu)}\tau_0^i$$

Varying (6.1) results in the following boundary conditions for the first two refined theories

$$t^{\alpha\beta}\nu_\alpha\nu_\beta - h\langle P_i\nu^i\rangle - \underline{k_{(\tau)}m_{\nu\tau}} = hb\frac{d}{ds}\langle P_k\tau^k\rangle - k_{(n)}T \| - k_{(n)}k_{(\nu)}M - \quad (6.9)$$
$$\frac{d}{ds}(k_{(\nu)}M_* + k_{(\tau)}M) - k_{(\tau)}\left(\frac{dM}{ds} + k_{(n)}M_*\right)$$
$$t^{\alpha\beta}\nu_\beta\tau_\alpha - h\langle P_i\tau^i\rangle + \underline{k_{(\nu)}m_{\nu\tau}} = hbk_{(n)}\langle P_i\tau^i\rangle + \frac{dT}{ds}\Big\| +$$
$$\frac{d}{ds}(k_{(\nu)}M) + k_{(\nu)}\left(\frac{dM}{ds} + k_{(n)}M_*\right) - k_{(n)}(k_{(\tau)}M + k_{(\nu)}M_*)$$
$$- m^{\alpha\beta}{}_{;\beta}\nu_\alpha - \frac{d}{ds}m_{\nu\tau} - h\langle P_i n^i\rangle = -hbk_{(\tau)}\langle P_k\tau^k\rangle -$$
$$\frac{d}{ds}\left(\frac{dM}{ds} + k_{(n)}M_*\right) + h^2\frac{d}{ds}\langle P_k\tau^k\zeta\rangle \Big| +$$
$$k_{(\tau)}(k_{(\tau)}M + k_{(\nu)}M_*)\| + k_{(\nu)}(T + k_{(\nu)}M)$$
$$m^{\alpha\beta}\nu_\alpha\nu_\beta + h^2\langle P_i\nu^i\zeta\rangle = hb\langle P_i n^i\rangle + \frac{d}{ds}M_* - k_{(n)}M$$

Here

$$m_{\nu\tau} = m^{\alpha\beta}\nu_\alpha\tau_\beta, \quad T = hbEA_{22}^* - \nu hb\langle P_i\nu^i\rangle$$
$$M = {}^1/_{12}\ h^3bEB_{22}^* + \nu h^2b\langle P_i\nu^i\zeta\rangle, \quad M_* = {}^1/_6h^3bD\omega_*$$

The classical terms are written in the left sides of (6.9) and the corrections in the right sides. The underlined terms and the terms behind the vertical bars can be discarded within the framework of the accuracy of the fundamental refined theory. Let us note that self-equilibration of the system of equations is lost here. Terms written after the double vertical bars are negligibly small in the theory taking account of geometric corrections.

The terms containing factors of the quantities b and $k_{(n)}$ have no analogs in classical theory. They are related to the fact that the boundary conditions are not posed at the points of external force application but at the site of a merger of the shell and boundary layer solutions. Additional moments and transverse forces hence occur.

The accuracy with which the functional I_1 is calculated does not permit writing boundary conditions in the theory taking account of shear. The relationships obtained for this theory can be used for closed shells.

7. Physically nonlinear theory. We replace the required functions (3.1). The y_α will be calculated exactly as in the physically linear theory if the stress-strain relations allow of linearization for strains of the order of εh_{**}^2. Then for y_α, as before, we obtain $y_\alpha \sim \varepsilon h_{**}$. Omitting further details, we formulate the result.

Let $U(\xi^\alpha, \zeta, h; \varepsilon_{\alpha\beta}; \varepsilon_{\alpha 3}; \varepsilon_{33})$ be a smooth function of all its arguments except ζ (piecewise smoothness in ζ is allowed), strictly convex in the strain tensor components and varying slightly in ξ^α at ranges on the order of h (the corresponding

scale l_U satisfies the condition $l_U \gg h$). These conditions are even satisfied in the limit as $h = 0$. Let us construct the longitudinal energy

$$U_{\parallel}(\xi^\alpha, \zeta; \varepsilon_{\alpha\beta}) = \min_{\varepsilon_{\alpha 3}, \varepsilon_{33}} U(\xi^\alpha, \zeta, 0; \varepsilon_{\alpha\beta}; \varepsilon_{\alpha 3}; \varepsilon_{33})$$

Let us calculate the function

$$\Phi(A, \rho) = \langle U_{\parallel}(\xi^\alpha, \zeta; A_{\alpha\beta} - h\rho_{\alpha\beta}\zeta)\rangle \tag{7.1}$$

The energy functional is given by (3.18) in a first approximation, where Φ should be understood to be the function (7.1) and $\rho_{\alpha\beta}$ the tensor (3.16)

8. On short-wave extrapolation. Construction of a two-dimensional theory of shells by asymptotic method consists of two logical steps: 1) the asymptotic analysis of the three-dimensional theory and the derivation of a two-dimensional theory of shells in the long wave region, and 2) extrapolation of the results to short waves. For brevity, the terminology used in dynamics is applied here and the slowly varying state of stress ($h / l \ll 1$) is called long-wave while the rapidly varying state of stress is short-wave.

The distinct asymptotic approaches in the first stage (analysis of the equations or energies) should result in identical results. In particular, the fundamental refined theory constructed above should agree with the refined theory proposed by Gol'denveizer [3]. Indeed, as Koiter and Heijden [16] showed, the Gol'denveizer theory allows an energy formulation with energies E of the form

$$E = \int_{\Omega_0} \Phi d\omega - \int_{\Gamma_0} \Psi' d\gamma, \quad \Psi' = \frac{h^3 b}{6}\left(1 - \frac{D}{2\mu}\right)\omega_*^2 \tag{8.1}$$

According to the fundamental refined theory, the energy has the form

$$E = \int_{\Omega_0} \Phi d\omega + \int_{\Gamma_*} \Psi d\gamma \tag{8.2}$$

It is seen from the formula (5.7) for Ψ that (8.1) and (8.2) agree within the framework of the accuracy under consideration.

The second step (extrapolation to short-waves is related to the desire to pose mathematically correct problems and, moreover, to apply the theory for "not very long" and "short" waves. Let us examine the correctness question.

The nature of boundary value problems which are correct for this system of equations is closely related to the type of system. The type of system is determined by its behavior at short waves. Hence, it is impossible to speak about the correctness and type of system of equations which would be deduced in its meaning in the long-wave region without preliminary extrapolation to short-waves.

We will call the extrapolation trivial if it consists simply of examining the constructed system of equations in which all the small terms, in the approximation under consideration, are discarded for both the long-wave and the short-wave states of stress. We will understand the extrapolation associated with the addition of a number of terms in the equations, which are negligible in the long-wave, but essential in the short-wave regions, as nontrivial.

It is clear that different nontrivial extrapolations are possible for the same system

of equations. This should not cause any perplexity since the short-wave states of stress do not allow of two-dimensional description and must be considered only in qualitative correspondence and, perhaps, a satisfactory description of the integral characteristics. The success of any nontrivial extrapolation from the viewpoint of the correctness of describing the state of stress is clarified in the solution of test problems.

In connection with the exposition, let us note that the question that is repeatedly evoked in the literature about whether the refined equations of shell dynamics should be equations of hyperbolic type is essentially a question about how to realize extrapolation of the equations derived for long-waves to short-waves. Both hyperbolic and non-hyperbolic extrapolation are allowable, however, as numerous computations [31] show, hyperbolic extrapolations describe the state of stress best on the whole.

The question of extrapolation as an independent question did not occur in the derivation of the classical theory of shells since the natural boundary value problems were correct after trivial extrapolation.

Trivial extrapolation of the refined theory of Gol'denveizer generates ambiguity of the solution, and even no solution for certain loads.

Let us explain this by the following example. Let us consider bending of a plate occupying the half-plane $x_1 \leqslant 0$. The edge $x_1 = 0$ is load-free. The classical boundary conditions have the form

$$m^{\alpha 1}_{,\alpha} + m^{12}_{,2} = 0, \quad m^{11} = 0 \tag{8.3}$$

where $m^{\alpha\beta}$ is the bending moments tensor. The refined boundary conditions [3, 16] contain a correction in the second equality in (8.3) (B is a constant)

$$m^{11} + Bhm^{12}_{,2} = 0 \tag{8.4}$$

It is assumed in (8.4) that the motion along the x^2, axis keeps the plate on the left.

The refined boundary conditions can be used by two methods: 1) to solve the problem with the boundary conditions (8.3), evaluate m^{12}, and find the correction to the solution, the plate bending by the external moment $(Bhm^{12})_{,2}$; 2) to solve the problem after a "trivial extrapolation", i.e., directly with the refined boundary conditions.

Let us examine the second method. For simplicity we put $\lambda = 0$, $2\mu = 1$ so that $m^{11} = u_{,11}$, $m^{12} = u_{,12}$, and $m^{22} = u_{,22}$. Let there be no external forces. The deflection u is the solution of the biharmonic equation $\Delta^2 u = 0$. We take a biharmonic function of the form $u = (a_1 + a_2x_1)\, e^{kx_1} \sin kx_2$. Substitution in the refined boundary conditions yields

$$ka_1 - a_2 = 0 \qquad (1 - Bhk)\, ka_1 + (2 - Bhk)\, a_2 = 0 \tag{8.5}$$

The determinant of the system of linear equations (8.5) vanishes for $k = 3 / (2Bh)$. Hence, there are nonzero solutions for zero external forces. Naturally these solutions possess high variability ($l \sim h$) and emerge beyond the framework of those states of stress for which the two-dimensional theory was constructed.

If external effects $P \sin (3x_2 / (2Bh))$ and $M \sin (3x_2 / (2Bh))$ are added to the right sides in (8.3) and (8.4), then the constants P and M can always be chosen such that the solution would not exist.

The question of whether the circumstance noted can affect the stability of a

numerical computation requires additional investigation. (In principle, the formulation of a problem with the boundary conditions (8.3) and (8.4) and additional rejection of solutions with high variability is apparently possible).

The linearized variant of the fundamental refined theory elucidated above can be considered as one of the possible methods of nontrivial extrapolation of the refined theory of Gol'denveizer to short waves. The uniqueness of the solution of problems in the linearized fundamental refined theory follows from the strict convexity of the energy, and the existence of the solution can be proved by standard calculus of variations methods.

The author is grateful to Le Khan' Xhau for verifying all the formulas and hence correcting a number of errors.

REFERENCES

1. Friedrichs, K. O., Kirchhoff's boundary conditions and the edge effect for elastic plates, Proc. Sympos. Appl. Math., Vol. 3, No. 4, 117—124, 1950.

2. Friedrichs, K. O. and Dressler, R. F., A boundary-layer theory for elastic plates, Communs. Pure and Appl. Math., Vol. 14, No. 1. 1961.

3. Gol'denveizer, A. L., Theory of Elastic Thin Shells, (English translation), Pergamon Press, Book No. 09561, 1961.

4. Green, A. E., Boundary-layer equations in the linear theory of thin elastic shells, Proc. Roy. Soc., London. Ser. A, Vol. 269, No. 1339, 481—491, 1962.

5. Vorovich, I. I., Certain results and problems of the asymptotic theory of plates and shells. Materials of the First All-Union School on the Theory and Numerical Methods of Analyzing Plates and Shells, Gegechkhori 1974. Tbilisi, Izd, Tbilisi Univ. 1975.

6. Koiter, W. T., A consistent first approximation in the general theory of thin elastic shells. Proc. Sympos. on the Theory of Thin Elastic Shells, Delft, 1959. Amsterdam, North-Holland Publ., 1960.

7. Novozhilov, V. V., Theory of Thin Shells, Sudpromgiz, Leningrad, 1962 (See also: Theory of Elasticity, (English translation), Pergamon Press, Book No. 09523, 1961).

8. John, F., Estimates for the derivatives of the stresses in thin shells and interior shell equations. Communs. Pure and Appl. Math., Vol. 18, No. 1—2, 1965.

9. Sanders, J. L., Jr., Nonlinear theories for thin shells. Quart. Appl. Math., Vol. 21, No. 1, 1963.

10. Naghdi, P. M., On the theory of thin elastic shells. Quart. Appl Math., Vol. 14, No. 4, 1957.

11. Mosolov, P. P., Asymptotic theory of thin rectilinear panels, Dokl. Akad. Nauk SSSR, Vol. 206, No. 2 1972.

12. Mosolov, P. P. and Miasnikov, V. P., Asymptotic theory of rigidly-plastic shells. PMM, Vol. 41, No. 3, 1977.

13. Shoikhet, B. A., On asymptotically exact equations of thin slabs of complex structure, PMM, Vol. 37, No. 5, 1973.

14. Shoikhet, B. A., An energy identity in physically nonlinear elasticity and error estimates of the plate equations, PMM, Vol. 40, No. 2, 1976.

15. Danielson, D. A., Improved error estimates in the linear theory of thin elastic shells, Proc. Koninkl. Nederl. Acad. Wet., B, Vol. 74, No. 3, 294–300, 1971.

16. Koiter, W. T. and Simmonds, J. G., Foundations of shell theory. IN: Theoretical and Applied Mechanics. Proc. 13th Internat. Congress Theor. and Applied Mech. (Moscow 1972). Berlin, Springer, 1973.

17. Aksentian, O. K. and Ustinov, Iu. A., Construction of refined applied theories for a plate on the basis of the equations of the theory of elasticity, PMM, Vol. 36, No. 2, 1972.

18. Heijden, A. M. A. van der, On Modified Boundary Conditions for the Free Edge of a Shell. Delft Univ. Press, 1976.

19. Heijden, A. M. A. van der. An asymptotic analysis of a plate with a circular hole, Lab. Eng'g Mech., Delft Univ. of Technol. Rept. 78.

20. Ladeveze, P., Justification de la théorie linearizé des coques elastiques, J. Mecanique, Vol. 15, No. 5, 1976.

21. Berdichevskii, V. L., On equations of the theory of anisotropic inhomogeneous rods. Dokl. Akad. Nauk SSSR, Vol. 228, No. 3, 1976.

22. Berdichevskii, V. L. and Kvashnina, S. S., On equations describing the transverse vibrations of elastic bars. PMM, Vol. 40, No. 1, 1976.

23. Berdichevskii, V. L., Variational-asymptotic methods. IN: Certain Questions of the Mechanics of a Continuous Medium. Moscow Univ. Press, 1978.

24. Berdichevskii, V. L., High-frequency long-wave vibrations of plates, Dokl. Akad. Nauk SSSR, Vol. 236, No. 6, 1977.

25. Kvashnina, S. S., High-frequency long-wave vibrations of elastic rods, PMM, Vol. 43, No. 2, 1979.

26. Berdichevskii, V. L., On averaging of periodic systems, PMM, Vol. 41, No. 6, 1977.

27. Sedov, L. I., Mechanics of a Continuous Medium , Vol. 2, 3d Ed., Moscow, "Nauka", 1976. (see also English translation, Pergamon Press, Book No. 09878, 1965).

28. Beckenbach, E. F. and Bellman, R., Inequalities, Moscow, "Mir", 1965.

29. Berdichevskii, V. L., On the dynamic theory of thin elastic plates. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No. 6, 1973.

30. Berdichevskii, V. L., Energy methods in certain problems of damping of solutions, PMM, Vol. 42, No. 1, 1978.

31. Grigoliuk, E. I. and Selezov, I. T., Nonclassical Theories of the Vibrations of Rods, Plates, and Shells, Moscow, VINITI (All Union Institute of Scientific and Technical Information), 1973.

Translated by M. D. F.